21.09.2020

II B.Sc Mathematics.

DIFFERENTIAL EQUATIONS

UNIT III Differential Equations of other types.

Simultaneous equations with constant coefficients – Total differential equations – Simultaneous Total differential equations – Equations of the form $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$.

I Total Differential Equations. [TDE]

In Total D.E, we have the differential coefficients of several dependent variables with reference to a single independent variable.

Definition: An ordinary D.E of the first order and first degree involving three variables is of the form

$$Pdx + Qdy + Rdz = 0 \rightarrow (1)$$

where P, Q, R are functions of x, y, z and x is the independent variable.

(ie) $P + Q\frac{dy}{dx} + R\frac{dz}{dx} = 0 \rightarrow (2)$

(1) is integrable only if

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0 \rightarrow (3)$$

Note: The condition given by (3) is said to be the necessary condition for the existence of the integral of equation (1).

Rule to solve $Pdx+Qdy+Rdz=0$.

- If the condition of integrability is satisfied, consider one of the variables, say: z, as constant.
- Consider $dz=0$.
- $z=$ constant $\Rightarrow dz=0$.
- Then integrate the equation $Pdx+Qdy=0$.
- Replace the arbitrary constant appearing in its integral by $\phi(z)$.
- Now differentiate the integral just obtained with respect to x, y, z.
- Finally compare the result with the given D.E. to determine $\phi(z)$.

Problems:

(1) (a) Solve $(y^2+yz)dx+(z^2+zx)dy+(y^2-xy)dz=0$.

Solution:

Given T.D.E. is

$$(y^2+yz)dx+(z^2+zx)dy+(y^2-xy)dz=0 \longrightarrow (1)$$

$$Pdx+Qdy+Rdz=0 \longrightarrow (2)$$

(1) is of the form which is a T.D.E.

Here $P=y^2+yz$; $Q=z^2+zx$; $R=y^2-xy$.

Step 1: To check integrability condition.

$P=y^2+yz$	$Q=z^2+zx$	$R=y^2-xy$.
$\Rightarrow \dfrac{\partial P}{\partial x}=0$	$\dfrac{\partial Q}{\partial x}=z$	$\dfrac{\partial R}{\partial x}=-y$
$\dfrac{\partial P}{\partial y}=2y+z$	$\dfrac{\partial Q}{\partial y}=0.$	$\dfrac{\partial R}{\partial y}=2y-x$
$\dfrac{\partial P}{\partial z}=y$	$\dfrac{\partial Q}{\partial z}=2z+x$	$\dfrac{\partial R}{\partial z}=0.$

Condition of integrability is

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0. \rightarrow (3)$$

L.H.S. of (3)

$$= P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right).$$

$$= (y^2+yz)\left[2z+x - (2y-x)\right] + (z^2+zx)\left[-y-y\right] + (y^2-xy)\left[2y+z-z\right].$$

$$= (y^2+yz)(2z-2y+2x) + (z^2+zx)(-2y) + (y^2-xy)(2y)$$

$$= \cancel{2zy^2} - \cancel{2y^3} + \cancel{2xy^2} + \cancel{2yz^2} + \cancel{2y^2z} + \cancel{2xyz}$$
$$- \cancel{2yz^2} - \cancel{2xyz} + \cancel{2y^3} - \cancel{2xy^2}.$$

$= 0.$ Condition is satisfied.

$\therefore$ Integrability condition exists.
$\Rightarrow$ Solution exists.

Step 2:
Consider $z = c$ (a constant) [We can assume any one of the variables as constant].

$\Rightarrow dz = 0.$

$\therefore$ (2) $\Rightarrow$ $P\,dx + Q\,dy = 0$

$\therefore$ (1) $\Rightarrow$ $(y^2+yz)\,dx + (z^2+zx)\,dy = 0.$

$\Rightarrow y(y+z)\,dx + z(z+x)\,dy = 0 \rightarrow (4)$

(4) is I order and I degree D.E. in variables x and y [with z as constant $z=c$].

$\therefore$ (4) $\Rightarrow$ $y(y+z)\,dx = -z(z+x)\,dy.$

$$\Rightarrow \frac{dx}{z(z+x)} = -\frac{dy}{y(y+z)}.$$

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$$\Rightarrow \frac{dx}{z(z+x)} + \frac{dy}{y(y+z)} = 0.$$

Since z is constant,

$$\frac{1}{z}\int\frac{dx}{x+z} + \cancel{\frac{1}{z}}\int\frac{dy}{y(y+z)} = 0 \longrightarrow (4)$$

$$\frac{1}{y(y+z)} = \frac{A}{y} + \frac{B}{y+z}$$

$$1 = A(y+z) + By.$$

Coefficient of $y \Rightarrow A+B=0.$

Constant $\Rightarrow$ $Az = 1$

$\Rightarrow A = 1/z$

$\Rightarrow B = -1/z.$

$$\Rightarrow \frac{1}{z}\int\frac{dx}{x+z} + \frac{1}{z}\left[\int\left(\frac{1}{y} - \frac{1}{y+z}\right)\right]dy = C.$$

$$\Rightarrow \frac{1}{z}\log(x+z) + \frac{1}{z}\left[\log y - \log(y+z)\right] = C.$$

$$\Rightarrow \log(x+z) + \log y - \log(y+z) = \text{constant}$$

$$\Rightarrow \log\frac{y(x+z)}{y+z} = \text{constant}.$$

$$\Rightarrow \frac{y(z+x)}{y+z} = \text{constant} = \phi(z) \longrightarrow (5)$$

$$\Rightarrow \boxed{y(z+x) - (y+z)\,\phi(z) = 0} \longrightarrow (6)$$

Differentiating w.r.t. x, y, z, we get

$$y(dz+dx) + (z+x)\,dy - (y+z)\,\phi'(z)\,dz - (dy+dz)\,\phi(z) = 0$$

$$\Rightarrow y\,dx + [z+x-\phi(z)]\,dy + [y-(y+z)\phi'(z)-\phi(z)]\,dz = 0. \longrightarrow (7).$$

Comparing (6) ~~and~~ (7) with the given D.E., we get

$$\underset{\text{I}}{\frac{y^2+yz}{y}} = \underset{\text{II}}{\frac{z^2+zx}{z+x-\phi(z)}} = \underset{\text{III}}{\frac{y^2-xy}{y-(y+z)\phi'(z)-\phi(z)}}.$$

[The Comparison of I & II reduces to (5) which gives no information about $\phi(z)$].

Take the ratios I & III, we get

(ie) $$\frac{y^2+yz}{y} = \frac{y^2-xy}{y-(y+z)\phi'(z)-\phi(z)}$$

$$y^2-xy = (y+z)\left[y-(y+z)\phi'(z)-\phi(z)\right]$$

$$= y^2+yz-(y+z)^2\phi'(z)-(y+z)\phi(z).$$

$$= y^2+yz-(y+z)^2\phi'(z)-y(z+x) \quad \text{since}$$

$$(5) \Rightarrow \frac{y(z+x)}{y+z} = \phi(z).$$

$$\Rightarrow \cancel{y^2}-\cancel{xy}-\cancel{y^2}-\cancel{yz}+\cancel{yz}+\cancel{xy} = (y+z)^2\phi'(z).$$

$$\Rightarrow (y+z)^2\phi'(z) = 0.$$

$$\Rightarrow \phi'(z) = 0 \Rightarrow \boxed{\phi(z) = c}$$

$\therefore$ The required solution is

$$\boxed{y(z+x) = (y+z)c.}$$

1(b) Solve $x\,dx + z\,dy + (y+2z)\,dz = 0$

Solution:

Given equation is

$$x\,dx + z\,dy + (y+2z)\,dz = 0.$$

Regrouping, $x\,dx + (y\,dz + z\,dy) + 2z\,dz = 0$ | $d(yz) = y\,dz + z\,dy$

Integrating, we get $\boxed{\frac{x^2}{2} + yz + z^2 = c}$

1(c) $(yz + xyz)\,dx + (zx + xyz)\,dy + (xy + xyz)\,dz = 0.$

Solution: Given equation is

$$(yz + xyz)\,dx + (zx + xyz)\,dy + (xy + xyz)\,dz = 0 \rightarrow (1)$$

Dividing (1) by xyz, we get

$$\frac{dx}{x} + dx + \frac{dy}{y} + dy + \frac{dz}{z} + dz = 0.$$

$$\Rightarrow \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} + dx + dy + dz = 0.$$

Integrating, we get

$$\log x + \log y + \log z + x + y + z = c$$

$$\Rightarrow \boxed{\log xyz + (x+y+z) = c}$$

1(d) $yz\,dx + 2zx\,dy - 3xy\,dz = 0.$

Solution:

Given equation is $yz\,dx + 2zx\,dy - 3xy\,dz = 0. \rightarrow (1)$

Dividing by xyz, we get

$$\frac{dx}{x} + 2\frac{dy}{y} - 3\frac{dz}{z} = 0.$$

Integrating, we get

$$\log x + 2\log y - 3\log z = \log c.$$

$$\Rightarrow \log x + \log y^2 - \log z^3 = \log c.$$

$$\Rightarrow \log \frac{xy^2}{z^3} = \log c \Rightarrow \frac{xy^2}{z^3} = c$$

$$\Rightarrow \boxed{xy^2 = cz^3}$$

$\times$

(1)(e) $(y^2+z^2-x^2)dx - 2xy\,dy - 2xz\,dz = 0.$

Solution: Given equation is

$$(y^2+z^2-x^2)dx - 2xy\,dy - 2xz\,dz = 0 \rightarrow (1)$$

Assume x as Constant.

$$\Rightarrow dx = 0.$$

$$\therefore (1) \rightarrow \Rightarrow -2xy\,dy - 2xz\,dz = 0.$$

$$\Rightarrow y\,dy + z\,dz = 0.$$

Integration $\Rightarrow \frac{y^2+z^2}{2} = c$

$$\Rightarrow y^2+z^2 = 2c = \phi(x).$$

$$\Rightarrow y^2+z^2 = \phi(x) \rightarrow (2)$$

Differentiating, we get

$$2y\,dy + 2z\,dz = \phi'(x)dx.$$

$$\Rightarrow \phi'(x)dx - 2y\,dy - 2z\,dz = 0 \rightarrow (3)$$

(1) & (3) $\Rightarrow \underbrace{\frac{\phi'(x)}{y^2+z^2-x^2}}_{\mathrm{I}} = \underbrace{\frac{-2y}{-2xy}}_{\mathrm{II}} = \underbrace{\frac{-2z}{-2xz}}_{\mathrm{III}}.$

Comparing I & II ratios, we get

$$\frac{\phi'(x)}{y^2+z^2-x^2} = \frac{1}{x}.$$

$$\Rightarrow \frac{\phi'(x)}{\phi(x)-x^2} = \frac{1}{x}.$$

$$\Rightarrow \phi'(x) = \frac{\phi(x)-x^2}{x}.$$

$$= \frac{\phi(x)}{x} - x.$$

$\phi(x) = v \Rightarrow \cancel{\phi'(x)} = v' = \frac{v}{x} - x.$

$\Rightarrow v' = \frac{v}{x} - x.$

$\Rightarrow \frac{dv}{dx} - \frac{v}{x} = -x \rightarrow (4)$

$P = -\frac{1}{x}$

$Q = -x$

$IF = e^{\int -\frac{1}{x} dx}$

$= e^{-\log x}$

$= e^{\log \frac{1}{x}}$

$\boxed{IF = \frac{1}{x}}$

Solution is

$v \cdot \left(\frac{1}{x}\right) = \int -x \left(\frac{1}{x}\right) dx + c.$

$\Rightarrow \frac{v}{x} = -\int dx + c.$

$\Rightarrow \frac{v}{x} = -x + c.$

$\Rightarrow \phi(x) = -x^2 + cx.$

$\Rightarrow \boxed{y^2 + z^2 + x^2 = cx}$

(ii) $yz\,dx + 2zx\,dy - 3xy\,dz = 0.$

Solution:

Given: $yz\,dx + 2zx\,dy - 3xy\,dz = 0 \rightarrow (1)$

$\div xyz \Rightarrow \frac{dx}{x} + 2\frac{dy}{y} - 3\frac{dz}{z} = 0.$

Integrating, we get

$\log x + 2\log y - 3\log z = \log c$

$\log x + \log y^2 - \log z^3 = \log c.$

$\Rightarrow \log \frac{xy^2}{z^3} = \log c.$

$\Rightarrow \frac{xy^2}{z^3} = c$

$\Rightarrow \boxed{xy^2 = cz^3}$

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1 (9) $(2xz - yz)\,dx + (2yz - zx)\,dy - (x^2 - xy + y^2)\,dz = 0.$

Solution:

Given $(2xz - yz)dx + (2yz - zx)\,dy - (x^2 - xy + y^2)dz = 0 \rightarrow (1)$

Assume $z = $ Constant.

$\Rightarrow dz = 0.$

$\therefore (1) \Rightarrow z(2x - y)\,dx + z(2y - x)\,dy = 0.$

$\Rightarrow (2x - y)\,dx + (2y - x)\,dy = 0.$

$\Rightarrow 2x\,dx - y\,dx + 2y\,dy - x\,dy = 0.$

$\Rightarrow 2x\,dx + 2y\,dy - (y\,dx + x\,dy) = 0.$

$\Rightarrow 2x\,dx + 2y\,dy - d(xy) = 0.$

Integrating, we get $\boxed{x^2 + y^2 - xy = c = \phi(z)}$

Differentiating we get

$2x\,dx + 2y\,dy - x\,dy - y\,dx = \phi'(z)\,dz.$

$2x\,dx + 2y\,dy - x\,dy - y\,dx - \phi'(z)dz = 0 \rightarrow (2)$

$\Rightarrow (2x - y)\,dx + (2y - x)\,dy - \phi'(z)dz$

$$(1) \& (2) \Rightarrow \underset{\text{I}}{\frac{2x - y}{z(2x - y)}} = \underset{\text{II}}{\frac{2y - x}{z(2y - x)}} = \underset{\text{III}}{\frac{\phi'(z)}{x^2 - xy + y^2}}.$$

Comparing I & III ratios,

$$\frac{1}{z} = \frac{\phi'(z)}{x^2 - xy + y^2} = \frac{\phi'(z)}{\phi(z)}.$$

$$\Rightarrow \frac{1}{z} = \frac{\phi'(z)}{\phi(z)} \Rightarrow \frac{\phi(z)}{z} = \phi'(z).$$

$$\Rightarrow \phi'(z) - \frac{\phi(z)}{z} = 0$$

$P = -1/z,\ Q = 0.$

$IF = e^{\int -\frac{1}{z}dz}$

$= \frac{1}{z}.$

$$\Rightarrow \phi(z) \cdot \frac{1}{z} = \int 0\,dz + c.$$

$$\Rightarrow \frac{\phi(z)}{z} = c \Rightarrow \phi(z) = cz.$$

$\therefore \boxed{x^2 + y^2 - xy = cz}$ is the required solution.

(1)(h) $yz\,dx + (xz - yz^3)\,dy - 2xy\,dz.$

~~(1)(i) $(xz - ny)\,dx + (nx$~~

(i) $(x+z)^2 dy + y^2 (dx + dz) = 0.$

Ans: $y(x+z) = c(x+y+z)$.

<u>Simultaneous Total Differential Equations.</u>

These equations in 3 variables are given by

$$\left.\begin{array}{l} P\,dx + Q\,dy + R\,dz = 0 \\ P'\,dx + Q'\,dy + R'\,dz = 0 \end{array}\right\} \longrightarrow (1)$$

where P, Q, R and P', Q', R' are any functions of x, y, z.

(a) If each of these equations is integrable and have solutions $\phi(x,y,z) = c$, $\psi(x,y,z) = c'$ respectively, then these taken together constitute the solution of the simultaneous equations (1).

(b) If one or both the equations (1) is not integrable, then we write these as follows:

$$\frac{dx}{QR' - Q'R} = \frac{dy}{RP' - R'P} = \frac{dz}{PQ' - P'Q}.$$

and solve these by the methods given below:

1. Method of Grouping.
2. Method of Multipliers.

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2.

Equations of the form $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$.

1. Method of Grouping.

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}.$$

$\frac{dx}{P} = \frac{dz}{R} \Rightarrow \phi(x, z) = c_1$, after integration $\to$ (1)

$\frac{dy}{Q} = \frac{dz}{R} \Rightarrow \psi(y, z) = c_2$, after integration, $\to$ (2)

(1) and (2) gives Complete solution.

Problems.

2(a) Solve $\frac{dx}{z^2 y} = \frac{dy}{z^2 x} = \frac{dz}{y^2 x}$.

Solution!

$$\frac{dx}{z^2 y} = \frac{dy}{z^2 x}$$

$$\Rightarrow \frac{dx}{y} = \frac{dy}{x}$$

$\Rightarrow x\,dx - y\,dy = 0$, $x^2 - y^2 = 2C$.

Integrating, we get $\boxed{x^2 - y^2 = c_1}$, $c_1 = 2c$. $\to$ (1)

$$\frac{dy}{z^2 x} = \frac{dz}{y^2 x},$$

$$\Rightarrow \frac{dy}{z^2} = \frac{dz}{y^2}$$

$$\Rightarrow y^2 dy = z^2 dz,$$

$$\Rightarrow \frac{y^3 - z^3}{3} = c'$$

$\Rightarrow \boxed{y^3 - z^3 = c_2}$, $c_2 = 3c'$. $\to$ (2)

(1) & (2) taken together gives the solution.

$$\boxed{x^2 - y^2 = c_1\,;\ y^3 - z^3 = c_2}$$

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(2)(b) $\frac{x\,dx}{y^2 z} = \frac{dy}{xz} = \frac{dz}{y^2}$.

Solution:

$\frac{x\,dx}{y^2 z} = \frac{dz}{y^2}$

$\Rightarrow x\,dx = z\,dz$

$\Rightarrow \frac{x^2}{2} - \frac{z^2}{2} = c$, on integration.

$\Rightarrow \boxed{x^2 - z^2 = c_1}$, $c_1 = 2c$.

$\frac{x\,dx}{y^2 z} = \frac{dy}{xz}$.

$\Rightarrow x^2 dx = y^2 dy$.

$\Rightarrow \frac{x^3}{3} - y^3/3 = c'$, on integration.

$\Rightarrow \boxed{x^3 - y^3 = c_2}$, $c_2 = 3c'$.

∴ The required solution is $\boxed{x^2 - z^2 = c_1;\ x^3 - y^3 = c_2}$

Method of multipliers.

By a choice of the multipliers l, m, n, we get

$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR}$ such that $lP + mQ + nR = 0$.

Then $l\,dx + m\,dy + n\,dz = 0$

Integrating, we get $\phi(x, y, z) = c_1 \rightarrow (1)$.

choosing another set of multipliers λ, μ, ν, we get

$\lambda P + \mu Q + \nu R = 0$.

$\Rightarrow \lambda\,dx + \mu\,dy + \nu\,dz = 0$.

Integrating, we get $\psi(x, y, z) = c_2 \rightarrow (2)$

(1) & (2) gives the required solution.

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Problems.

(3) (a) Solve $\dfrac{dx}{x(y^2-z^2)} = \dfrac{dy}{-y(z^2+x^2)} = \dfrac{dz}{z(x^2+y^2)}$.

Solution: Given $\dfrac{dx}{x(y^2-z^2)} = \dfrac{dy}{-y(z^2+x^2)} = \dfrac{dz}{z(x^2+y^2)}$.

Using multipliers x, y, z, we get

$$\frac{x\,dx + y\,dy + z\,dz}{x^2(y^2-z^2) - y^2(z^2+x^2) + z^2(x^2+y^2)} = \frac{x\,dx + y\,dy + z\,dz}{x^2y^2 - x^2z^2 - y^2z^2 - y^2x^2 + z^2x^2 + z^2y^2}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{0},$$

$\Rightarrow x\,dx + y\,dy + z\,dz = 0.$

Integrating, we get $\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = C \Rightarrow \frac{x^2}{\cancel{2}} + y^2 + z^2 = 2C$,

$\therefore x^2 + y^2 + z^2 = C_1 \rightarrow$ (1)

Again using multipliers $\frac{1}{x}, -\frac{1}{y}, -\frac{1}{z}$, we get

$$\frac{\frac{1}{x}dx - \frac{1}{y}dy - \frac{1}{z}dz}{y^2 - z^2 + z^2 + x^2 - x^2 - y^2} = \frac{\frac{1}{x}dx - \frac{1}{y}dy - \frac{1}{z}dz}{0},$$

$\Rightarrow \frac{dx}{x} - \frac{dy}{y} - \frac{dz}{z} = 0.$

Integrating, we get

$\log x - \log y - \log z = \log c' \Rightarrow \log \frac{x}{yz} = \log c'$

$\Rightarrow \frac{x}{yz} = c' \Rightarrow x = c_2 yz \rightarrow$ (2)

(or) $yz = xc'$

$\therefore$ The required solution is

$$\boxed{x^2 + y^2 + z^2 = c_1 ;\ x = c_2 yz}$$

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(3) (c) Solve $\frac{dx}{yz} = \frac{dy}{xz} = \frac{dz}{xy}$.

Solution: Given $\frac{dx}{yz} = \frac{dy}{xz} = \frac{dz}{xy}$.

$\frac{dx}{yz} = \frac{dy}{xz}$

$\Rightarrow \frac{dx}{y} = \frac{dy}{x}$

$\Rightarrow x\,dx = y\,dy$

Integrating, we get

$\frac{x^2}{2} = \frac{y^2}{2} + c$

$\Rightarrow x^2 - y^2 = 2c = c_1$

$\Rightarrow \boxed{x^2 - y^2 = c_1}$

$\frac{dx}{yz} = \frac{dz}{xy}$.

$\Rightarrow \frac{dx}{z} = \frac{dz}{x}$.

$\Rightarrow x\,dx = z\,dz$

Integrating, we get

$x^2/2 = z^2/2 + c'$

$\Rightarrow x^2 - z^2 = 2c' = c_2$

$\Rightarrow \boxed{x^2 - z^2 = c_2}$

$\therefore$ The required solution is $\boxed{x^2 - y^2 = c_1;\ x^2 - z^2 = c_2}$

(2) (d) $\frac{dx}{xz} = \frac{dy}{yz} = \frac{2dz}{(x+y)^2}$.

Solution: $\frac{dx}{xz} = \frac{dy}{yz}$

$\Rightarrow \frac{dx}{x} = \frac{dy}{y}$.

$\Rightarrow \log x = \log y + \log c$

$\Rightarrow \log x - \log y = \log c_1$

$\Rightarrow \frac{x}{y} = c_1$.

$\Rightarrow \boxed{x = c_1 y}$

$\frac{dx+dy}{z(x+y)} = \frac{2dz}{(x+y)^2}$

$\Rightarrow dx+dy = \frac{2z\,dz}{x+y}$

$\Rightarrow (x+y)(dx+dy) = 2z\,dz$

$\Rightarrow \frac{(x+y)^2}{2} = z^2 + c'$

$\Rightarrow (x+y)^2 - 2z^2 = 2c' = c_2$.

$\boxed{(x+y)^2 - 2z^2 = c_2}$

$\therefore$ The required solution is $x = c_1 y;\ (x+y)^2 - 2z^2 = c_2$

3 (b) Solve $\frac{dx}{-y^2-z^2} = \frac{dy}{xy} = \frac{dz}{xz}$ → (1)

Solution:

$\frac{dy}{xy} = \frac{dz}{xz}$

$\Rightarrow \frac{dy}{y} = \frac{dz}{z}$

Integrating we get

$\log y = \log z + \log c_1$

$\Rightarrow \boxed{\frac{y}{z} = c_1}$

$\Rightarrow \boxed{y = c_1 z}$

Each of the ratios in (1)

$\frac{dx}{-y^2-z^2} = \frac{dy}{xy} = \frac{dz}{xz} = \frac{xdx + ydy + zdz}{x(-y^2-z^2) + xy^2 + xz^2}$

$= \frac{xdx + ydy + zdz}{0}$

$\Rightarrow xdx + ydy + zdz = 0.$

Integrating, we get $x^2 + y^2 + z^2 = 2c'$

$\Rightarrow \boxed{x^2 + y^2 + z^2 = c_2}$

∴ The general solution is $\phi\left(\frac{y}{z},\ x^2+y^2+z^2\right) = 0.$

3 (c) $\frac{dx}{y-zx} = \frac{dy}{yz+x} = \frac{dz}{x^2+y^2}.$

Solution: Each ratio $= \frac{ydx + xdy}{y(y-zx) + x(yz+x)}$

$= \frac{ydx + xdy}{x^2+y^2}$

Now $\frac{ydx + xdy}{x^2+y^2} = \frac{dz}{x^2+y^2}$ $\Rightarrow ydx + xdy = dz$

$\Rightarrow d(xy) = dz$

Integrating, we get $\boxed{xy = z + c_1}$

using multipliers x, y, z, we get

$\frac{xdx - ydy + zdz}{x(y-zx) - y(yz+x) + z(x^2+y^2)} = \frac{xdx - ydy + zdz}{0}$

$\Rightarrow xdx - ydy + zdz = 0 \Rightarrow \boxed{x^2 - y^2 + z^2 = 2c' = c_2}$

∴ The general solution is $\phi(xy - z,\ x^2 - y^2 + z^2) = 0.$

3 (d) Solve $\frac{dx}{xy} = \frac{dy}{y^2} = \frac{dz}{x(yz-2x)}$.

Solution:

$\frac{dx}{xy} = \frac{dy}{y^2}$

$\Rightarrow \frac{dx}{x} = \frac{dy}{y}$

$\Rightarrow \log x = \log y + \log c_1$

$\Rightarrow \boxed{x = c_1 y}$

Put $x = c_1 y$ in second and third ratios we get

$\frac{dy}{y^2} = \frac{dz}{c_1 y(yz - 2c_1 \cancel{y})}$

$\Rightarrow \frac{dy}{y^2} = \frac{dz}{c_1 y^2(z-2c_1)}$.

$\Rightarrow c_1\, dy = \frac{dz}{z-2c_1}$.

$\Rightarrow c_1 y = \log(z-2c_1) + \log c_2$

$\Rightarrow e^{c_1 y} = c_2(z-2c_1)$

$\Rightarrow e^{x} = c_2\left(z - \frac{2x}{y}\right)$.

$\Rightarrow \boxed{\frac{ye^{x}}{yz-2x} = c_2}$.

$\therefore$ The general solution is

$\phi\left(\frac{x}{y}, \frac{ye^{x}}{yz-2x}\right) = 0$.

3 (e) Solve $\frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx}$ [Ans. $lx+my+nz = c_1$, $x^2+y^2+z^2 = c_2$]

(f) $\frac{dx}{x^2-yz} = \frac{dy}{y^2-zx} = \frac{dz}{z^2-xy}$ [Ans. $\frac{x-y}{y-z} = c_1$, $\frac{z-x}{y-z} = c_2$]

(g) $\frac{dx}{x(y^2-z^2)} = \frac{dy}{y(z^2-x^2)} = \frac{dz}{z(x^2-y^2)}$ [Ans. $xyz = c_1$, $x^2+y^2+z^2 = c_2$]

(h) $\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2xz}$.

[Ans. $y = c_1 z$, $x^2+y^2+z^2 = c_2 z$].

II BSc Mathematics.

DIFFERENTIAL EQUATIONS

UNIT III - Continued....

Simultaneous D.E. with Constant Coefficients.

Problems.

(1) Solve $\frac{dx}{dt} + y = e^t$; $x - \frac{dy}{dt} = t$.

Solution: Given $\frac{dx}{dt} + y = e^t \rightarrow (1)$, $x - \frac{dy}{dt} = t \rightarrow (2)$

$\Rightarrow Dx + y = e^t \rightarrow (1)$

& $x - Dy = t \rightarrow (2)$.

$(1) \times D \Rightarrow D^2x + \cancel{Dy} = D(e^t) = e^t \rightarrow (3)$

$(2) \Rightarrow x - \cancel{Dy} = t$

$(3) + (2) \Rightarrow D^2x + x = e^t + t$

$\Rightarrow (D^2+1)x = e^t + t \rightarrow (4)$.

To find C.F. Consider $(D^2+1)x = 0$.

A.E. is $m^2+1=0$

$\boxed{m = \pm i} = \alpha \pm i\beta$

$\Rightarrow \alpha = 0, \beta = 1$.

C.F $= e^{\alpha t}(A\cos\beta t + B\sin\beta t)$

$= e^{0t}(A\cos t + B\sin t$.

$\therefore \boxed{\text{C.F.} = A\cos t + B\sin t}$

To find P.I.

$$P.I. = \frac{1}{D^2+1}(e^t + t)$$

$$= \frac{1}{D^2+1}e^t + \frac{1}{D^2+1}.t \longrightarrow (5)$$

$$= P.I._1 + P.I._2.$$

$$P.I._1 = \frac{1}{D^2+1}.e^t \quad (D = a = 1)$$

$$= \frac{1}{1+1}e^t = \frac{1}{2}e^t$$

$$\therefore \boxed{P.I_1 = \frac{1}{2}e^t}$$

$$P.I._2 = \frac{1}{D^2+1}t.$$

$$= (1+D^2)^{-1}t.$$

$$= (1-D^2)t$$

$$= t.$$

$$\Rightarrow \boxed{P.I._2 = t}$$

$\therefore$ General Solution is

$$\cancel{y} \; x = C.F. + P.I_1 + P.I_2$$

$$\Rightarrow \boxed{x = A\cos t + B\sin t + \frac{e^t}{2} + t}$$

Now (1) $\Rightarrow y = e^t - D(x)$

$$= e^t - \left[-A\sin t + B\cos t + \frac{e^t}{2} + 1\right].$$

$$\Rightarrow \boxed{y = A\sin t - B\cos t + \frac{e^t}{2} - 1}$$

② Solve the Simultaneous equations

$\frac{d^2x}{dt^2} - 3x - 4y = 0$; $\frac{d^2y}{dt^2} + x + y = 0$.

Solution:

Given equations are

$D = \frac{d}{dt}$

$D^2 = \frac{d^2}{dt^2}$

$D^2x - 3x - 4y = 0$

$\Rightarrow (D^2-3)x - 4y = 0 \longrightarrow ①$

& $D^2y + x + y = 0$

$\Rightarrow x + (D^2+1)y = 0 \longrightarrow ②$.

① $\times (D^2+1) \Rightarrow (D^2-3)(D^2+1)x - \cancel{4(D^2+1)y} = 0 \longrightarrow ③$

② $\times 4 \Rightarrow 4x \quad + \cancel{4(D^2+1)y} = 0 \longrightarrow ④$

③ + ④ $\Rightarrow (D^2-3)(D^2+1)x + 4x = 0$.

$\Rightarrow (D^4 + D^2 - 3D^2 - 3 + 4)x = 0$

$\Rightarrow (D^4 - 2D^2 + 1)x = 0$.

A.E. is $m^4 - 2m^2 + 1 = 0$

$\Rightarrow (m^2-1)^2 = 0$

$\Rightarrow m^2 - 1 = 0; \quad m^2 - 1 = 0$

$\Rightarrow m = \pm 1, \quad m = \pm 1$.

$\Rightarrow m = -1, -1, +1, +1$.

$cf = (At+B)e^{m_1 t} + (Ct+D)(e^{m_2 t})$

$\Rightarrow cf = (At+B)e^{t} + (ct+D)e^{-t}$, P.I. = 0.

Now $x =$ C.F. + P.I

$\Rightarrow \boxed{x = (At+B)e^{t} + (ct+D)e^{-t}}$

$(1) \Rightarrow D^2x - 3x - 4y = 0.$

$\Rightarrow 4y = D^2x - 3x \longrightarrow (5)$

Now $x = (At+B)e^{t} + (Ct+D)e^{-t}$.

$= Ate^{t} + Be^{t} + Cte^{-t} + De^{-t}$.

$\Rightarrow Dx = A(te^{t}+e^{t}) + Be^{t} + C(-te^{-t}+e^{-t}) + De^{-t}(-1)$

$= A(te^{t}+e^{t}) + Be^{t} + C(-te^{-t}+e^{-t}) - De^{-t}$.

$\& \; D^2x = A[te^{t}+e^{t}+e^{t}) + Be^{t} + C(te^{-t} - e^{-t} - e^{-t}) + De^{-t}$.

$= A(te^{t}+2e^{t}) + Be^{t} + C(te^{-t} - 2e^{-t}) + De^{-t}$.

$\therefore (5) \Rightarrow 4y = A(te^{t}+2e^{t}) + Be^{t} + C(te^{-t} - 2e^{-t}) + De^{-t}$

$- 3(Ate^{t} + Be^{t} + Cte^{-t} + De^{-t})$.

$= A[te^{t} + 2e^{t} - 3te^{t}] + B(e^{t} - 3e^{t}) +$

$C[te^{-t} - 2e^{-t} - 3te^{-t}] + D(e^{-t} - 3e^{-t})$.

$4y = A(2e^{t} - 2te^{t}) - 2Be^{t} + C(-2e^{-t} - 2te^{-t})$

$- 2De^{-t}$.

$$\boxed{y = \frac{1}{2}\left[A(e^{t} - te^{t}) - Be^{t} - C(e^{-t} - te^{-t}) - De^{-t}\right]}$$

$x \; \& \; y$ gives the general solution.

3. Solve the simultaneous equations

$\frac{dx}{dt} + 5x - 2y = t$; $\frac{dy}{dt} + 2x + y = 0$ given that $x = 0, y = 0$ when $t = 0$.

Solution:

Given $\frac{dx}{dt} + 5x - 2y = t$

$\Rightarrow Dx + 5x - 2y = t$

$\Rightarrow (D+5)x - 2y = t \rightarrow (1)$

& $\frac{dy}{dt} + 2x + y = 0.$

$\Rightarrow (D+1)y + 2x = 0$

$\Rightarrow 2x + (D+1)y = 0 \rightarrow (2)$

(1) $\times (D+1) \Rightarrow (D+1)(D+5)x - 2(D+1)y = (D+1)t.$

(2) $\times 2 \Rightarrow 4x + 2(D+1)y = 0.$

Adding $\Rightarrow [(D+1)(D+5) + 4]x = (D+1)t.$

$\Rightarrow (D^2 + 5D + D + 5 + 4)x = t + 1.$

$\Rightarrow (D^2 + 6D + 9)x = t + 1 \rightarrow (3)$

To find C.F

Consider $(D^2 + 6D + 9)x = 0.$

A E $\Rightarrow m^2 + 6m + 9 = 0$

$\Rightarrow (m+3)^2 = 0 \Rightarrow m = -3, -3$

$$\boxed{C.F. = (At + B)e^{-3t}}$$

<u>To find P.I</u>

$$P.I = \frac{1}{D^2+6D+9}(t+1)$$

$$= \frac{1}{D^2+6D+9}t + \frac{1}{D^2+6D+9}.e^{0t} \rightarrow (4)$$

$$= P.I_1 + P.I_2.$$

$$P.I_2 = \frac{1}{D^2+6D+9}e^{0t} \quad \boxed{D = a = 0}$$

$$= \frac{1}{9}e^{0t} = \frac{1}{9}$$

$$\Rightarrow \boxed{P.I_2 = \frac{1}{9}}$$

$$P.I._1 = \frac{1}{D^2+6D+9}.t.$$

$$= \frac{1}{9\left[1+\left(\frac{D^2+6D}{9}\right)\right]}t.$$

$$= \frac{1}{9}\left[1+\left(\frac{D^2+6D}{9}\right)\right]^{-1}(t).$$

$$= \frac{1}{9}\left[1-\left(\frac{D^2+6D}{9}\right)\right]t.$$

$$= \frac{1}{9}\left[t - \frac{D^2}{9}(t) - \frac{6D}{9}(t)\right]$$

$$= \frac{1}{9}\left[t - 0 - \frac{2D(t)}{3}\right].$$

$$= \frac{1}{9}\left(t - 2/3\right).$$

$$\boxed{P.I_1 = \frac{t}{9} - \frac{2}{27}}$$

$\therefore \quad x = C.F. + P.I_1 + P.I_2.$

$\Rightarrow$ General solution is

$$x = (At+B)e^{-3t} + \frac{1}{9} + \frac{t}{9} - \frac{2}{27}.$$

$$\Rightarrow \boxed{x = (At+B)e^{-3t} + \frac{t}{9} - \frac{1}{27}}$$

Now ① $\Rightarrow \quad \frac{dx}{dt} + 5x - 2y = t.$

$$\Rightarrow \quad 2y = \frac{dx}{dt} + 5x - t.$$

$$\Rightarrow 2y = D\left[Ate^{-3t} + Be^{-3t} + \frac{t}{9} - \frac{1}{27}\right] + 5\left[(At+B)e^{-3t} + \frac{t}{9} - \frac{1}{27}\right] - t.$$

$$= Ae^{-3t} - 3Ate^{-3t} - 3Be^{-3t} + \frac{1}{9} + 5(At+B)e^{-3t} + \frac{5t}{9} - \frac{5}{27} - t.$$

$$= Ae^{-3t} - 3Ate^{-3t} - 3Be^{-3t} + \frac{1}{9} + 5Ate^{-3t} + 5Be^{-3t} + \frac{5t}{9} - \frac{5}{27} - t.$$

$$= 2Ate^{-3t} + Ae^{-3t} + 2Be^{-3t} + \frac{5t-9t}{9} + \frac{3-5}{27}.$$

$$\Rightarrow 2y = 2Ate^{-3t} + Ae^{-3t} + 2Be^{-3t} - \frac{4t}{9} - \frac{2}{27}.$$

$$\Rightarrow \boxed{y = Ate^{-3t} + \frac{Ae^{-3t}}{2} + Be^{-3t} - \frac{2t}{9} - \frac{1}{27}.}$$